Peter Lee on October 13, 2009

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<u>Overall strategy.</u> 1. Given a Lie BiAlg A embed it into a QTLBA DA

2. Quantize DA to get a BA H.

3. Find within H a sub-BA H_+ which quantizes A.

<u>Step 2</u> Assume give a monoidal structure on Rep A (maybe braided, maybe with duals) $\longrightarrow$ extract a co-product on A.

Reminder on monoidal functors: Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between monoidal categories $\mathcal{C}$ & $\mathcal{D}$. A tensor structure on F is

1. Natural isomorphisms $J_{xy}: F(x) \otimes F(y) \to F(x \otimes y)$ where $x, y \in Ob(\mathcal{C})$
2. $j: 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$

Such that

$$\begin{tikzcd}
(F(x)\otimes F(y))\otimes F(z) \arrow[r, "J\otimes 1"] \arrow[d, "\Phi"] & F(x\otimes y)\otimes F(z) \arrow[r, "J"] \arrow[d, phantom, "\circlearrowleft"] & F((x\otimes y)\otimes z) \arrow[d, "F(\Phi)"] \\
F(x)\otimes(F(y)\otimes F(z)) \arrow[r, "1\otimes J"] & F(x)\otimes F(y\otimes z) \arrow[r, "J"] & F(x\otimes(y\otimes z))
\end{tikzcd}$$

(and another one involving j)

Let A be a bi-alg, then rep A has a monoidal structure. denoted $\text{Rep}_{\Delta\epsilon} A$.

<u>Thm</u> Let A be an algebra. A monoidal structure $\mathcal{C}$ on Rep(A) plus a tensor structure on Forget: Rep $A \to$ Vect induces (Δ, ϵ) on A s.t. $\text{Rep}_{\Delta,\epsilon} A \cong \mathcal{C}$.

Pf/construction Recall that $A \cong \mathrm{End}(F)$.

Consider the bifunctor $F^2 : \mathcal{C} \times \mathcal{C} \to \mathrm{Vect}$ by

$$(M,N) \to {}_0M \otimes {}_0N$$

Claim $\mathrm{End}(F^2) = (\mathrm{End}\, F)^{\otimes 2}$

Then $A = \mathrm{End}\, F \to \mathrm{End}(F^2) = (\mathrm{End}\, F)^{\otimes 2} = A^{\otimes 2}$

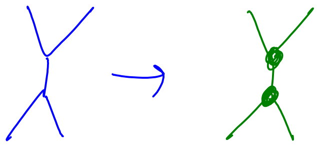
This map is (for $a \in A$)

$${}_0M \otimes {}_0N \xrightarrow{J} {}_0(M \otimes N) \xrightarrow{a} {}_0(M \otimes N) \xrightarrow{J^{-1}} {}_0M \otimes {}_0N$$

Pf of claim The inclusion $\supset$ is easy.

Now let $\phi \in \mathrm{End}(F^2)$. Let $\alpha = \phi_{AA}(1 \otimes 1) \in A \otimes A$....

Point to consider: Interpret this as



Example Let G be a finite group, $\mathcal{F}(G) = \{f : G \to \mathbb{C}\}$ is a BA with

$$m : \mathcal{F}(G) \otimes \mathcal{F}(G) \cong \mathcal{F}(G \times G) \xrightarrow{m} \mathcal{F}(G)$$

by declaring $(mf)(a) = f(a,a)$

$$(\Delta f)(a,b) = f(a \cdot b)$$

$$\eta : \mathbb{C} \to \mathcal{F}(G) \text{ by } \eta(1) \equiv 1$$

$$\epsilon(f) = f(1_G)$$

Problem: Develop a "KTG" picture for this example?

Let $\mathcal{C}$ be the category of 1D reps of $\mathcal{F}(G)$

So $\phi \in \mathcal{C}$ is really a map $\phi : \mathcal{F}(G) \to \mathbb{C}$,

so G can be viewed as a basis for $\mathcal{C}$.

and then $g \otimes h = gh$

In this context, Φ an associator is an isomorphism

$$\phi_{gfh} : (g \otimes f) \otimes h \longrightarrow g \otimes (f \otimes h)$$

So Φ_{gfh} is some scalar so $\Phi : G \times G \times G \to \mathbb{C}^*$

pentagon:

$$\Phi(a,b,c) \cdot \Phi(a,bc,d)\, \Phi(b,c,d) / \Phi(a,b,cd)^{-1} \Phi(ab,c,d)^{-1} = 1$$

So $\Phi \in Z^3(G, \mathbb{C}^*)$

Q When is $\mathcal{C}_\Phi \cong \mathcal{C}_{\Phi'}$?

Let $F : \mathcal{C}_\Phi \to \mathcal{C}_{\Phi'}$ be trivial on objects. Does F have a tensor structure?

$$\forall g, h \in G \quad \exists ? \quad J_{gh} : F(g) \otimes F(h) \to F(gh)$$

(i.e., a scalar)

s.t.

$$\Phi(g,h,f)\, J(h,f)\, J(g,hf) = J(g,h)\, J(gh,f)\, \Phi'(g,hf)$$

or

$$\frac{\Phi(g,h,f)}{\Phi'(g,h,f)} = J_{gh,f}\, J_{g,h}\, J_{h,f}^{-1}\, J_{g,hf} \; (dJ)(g,h,f)$$

The family of tensor structures on F

$$\Longleftrightarrow J \in C^2(G, \mathbb{C}^*) \text{ s.t. } dJ = \Phi/\Phi'$$